The Correlated Binomial Distribution - Part I

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April 2012

In August of 2004 Moody’s published "The Moody’s Correlated Binomial Default Distribution”. Whereas the mathematics can be applied to any binomial event distribution where events are correlated Moody’s emphasis was on modeling correlated defaults. In structured finance correlation is critical. Because the Moody’s publication was sparse on the mathematical derivation of the correlated binomial this paper goes through the derivation step by step. As is usually the case we will start with a hypothetical problem, derive and discuss the mathematics needed to solve that problem and then apply the mathematics so actually solving the problem. The problem that we will be solving is the problem presented in Part I but with a correlation assumption. Let’s begin...

Our Hypothetical Problem

In the PDF on The Binomial Distribution we considered the following problem...

Imagine that we have a bond portfolio that contains three bonds each with a principal balance of $1,000,000 and each with a maturity date one year hence. If at maturity the bond does not default we will receive the principal balance plus 15.00% simple interest. If at maturity the bond does default we will receive the recovery on that bond which is expected to be 40% of the principal balance. Each bond has an annual default probability of 0.10 and defaults are independent.

Question 1: What is the probability that our bond portfolio will experience zero defaults, one default, two defaults and three defaults?

Question 2: How much principal and interest can we expect to collect in one years time and what is our expected total return?

Given independent defaults the answer to the problem was...

<table>
<thead>
<tr>
<th>Default</th>
<th>No Default</th>
<th>Combin</th>
<th>Indiv Prob</th>
<th>Total Prob</th>
<th>Payoff</th>
<th>Expected</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0.7290</td>
<td>0.7290</td>
<td>3,450,000</td>
<td>2,515,050</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0.0810</td>
<td>0.2430</td>
<td>2,700,000</td>
<td>656,100</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>0.0090</td>
<td>0.0270</td>
<td>1,950,000</td>
<td>52,650</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0.0010</td>
<td>0.0010</td>
<td>1,200,000</td>
<td>1,200</td>
</tr>
<tr>
<td>Total</td>
<td>8</td>
<td>8</td>
<td>1.0000</td>
<td>3,225,000</td>
<td>3,225,000</td>
<td></td>
</tr>
</tbody>
</table>

Total expected return = \frac{3,225,000}{3,000,000} - 1 = 7.50%

In Part II we will consider the case where defaults are correlated, which for credit risk problems is much more realistic than assuming independence. Our Part II task is to answer the hypothetical problem above assuming that default correlation is 0.30 rather then zero (i.e. independent defaults).

Setting Up The Problem

We will use the variable $B_1$ to represent the first bond in our portfolio, $B_2$ to represent the second bond in our portfolio and $B_3$ to represent the third bond in our portfolio. Note that order is not important. The variable $X_1$ represents the default indicator applicable to bond $B_1$ and will take the value of one if bond $B_1$ defaults and zero if bond $B_1$ does not default. The variable $X_2$ represents the default indicator applicable to bond $B_2$ and will take the
value of one if bond $B_2$ defaults and zero if bond $B_2$ does not default. The variable $X_3$ represents the default indicator applicable to bond $B_3$ and will take the value of one if bond $B_3$ defaults and zero if bond $B_3$ does not default.

Per the hypothetical problem above the unconditional probability of default is...

$$p_0 = 0.10$$

(1)

There Have Been No Prior Bond Defaults

The equation for the conditional default correlation between $X_1$ and $X_2$ given no prior defaults is...

$$\theta_{1,2} = \frac{E[X_1 X_2] - E[X_1] E[X_2]}{\sqrt{Var(X_1) Var(X_2)}}$$

(2)

The expected value of $X_1$ is...

$$E[X_1] = (0 \times P[X_1 = 0]) + (1 \times P[X_1 = 1])$$
$$= P[X_1 = 1]$$
$$= p_0$$

(3)

Using the mathematics of Equation (3) it can be shown that the expected values of $X_2$ and $X_3$ are...

$$E[X_2] = p_0$$
$$E[X_3] = p_0$$

(4)

(5)

The expected value of the square of $X_1$ is...

$$E[X_1^2] = (0^2 \times P[X_1 = 0]) + (1^2 \times P[X_1 = 1])$$
$$= P[X_1 = 1]$$
$$= p_0$$

(6)

Using the mathematics of Equation (6) it can be shown that the expected values of the squares of $X_2$ and $X_3$ are...

$$E[X_2^2] = p_0$$
$$E[X_3^2] = p_0$$

(7)

(8)

The variance is defined as the second moment of the distribution of $X$ minus the square of the first moment of the distribution of $X$. Using Equations (3) and (6) above the variance of $X_1$ is...

$$Var(X_1) = E[X_1^2] - (E[X_1])^2$$
$$= p_0 - p_0^2$$
$$= p_0 (1 - p_0)$$

(9)

Using the mathematics of Equation (9) it can be shown that the variances of $X_2$ and $X_3$ are...

$$Var(X_2) = p_0 (1 - p_0)$$
$$Var(X_3) = p_0 (1 - p_0)$$

(10)

(11)

The expected value of the product of $X_1$ and $X_2$ is...

$$E[X_1 X_2] = (0 \times 0 \times P[X_1 = 0 \cap X_2 = 0]) + (1 \times 0 \times P[X_1 = 1 \cap X_2 = 0]) +$$
$$P[X_1 = 1 \cap X_2 = 1]$$
$$= p_0 (1 - p_0)$$

(12)

Note that per Equation (12) above the expected value of the product of $X_1$ and $X_2$ is the probability that bonds $B_1$ and $B_2$ both default. The equation for this compound probability is the probability that bond $B_1$ defaults times...
the probability that bond $B_2$ defaults given that bond $B_1$ defaults.

We can equate equations Equations (2) and (12) and solve for the conditional probability that bond $B_2$ will default given that bond $B_1$ has already defaulted. The equation for the conditional probability that $X_2 = 1$ given that $X_1 = 1$ is...

$$
\begin{align*}
E[X_1 X_2] &= \theta_{1,2} \sqrt{Var(X_1)} Var(X_2) + E[X_1] E[X_2] \\
P[X_1 = 1 | P[X_2 = 1 | X_1 = 1] &= \theta_{1,2} \sqrt{p_0(1 - p_0) p_0(1 - p_0) + p_0^2} \\
p_0 P[X_2 = 1 | X_1 = 1] &= \theta_{1,2} p (1 - p_0) + p_0 \\
P[X_2 = 1 | X_1 = 1] &= \theta_{1,2} (1 - p - 0) + p_0
\end{align*}
$$
(13)

For the sake of convenience we will define the variable $p_1$ to be the conditional probability that bond $B_2$ defaults given that bond $B_1$ has already defaulted. Using this definition we can rewrite Equation (13) above as...

$$
p_1 = P[X_2 = 1 | X_1 = 1] \\
= \theta_{1,2} (1 - p_0) + p_0 \\
= p_0 (1 - \theta_{1,2}) + \theta_{1,2}
$$
(14)

We can now combine Equations (12) and (14) and show that the expected value of the product of $X_1$ and $X_2$ can be rewritten is...

$$
E[X_1 X_2] = P[X_1 = 1] P[X_2 = 1 | X_1 = 1] \\
= p_0 p_1
$$
(15)

Using the mathematics of Equation (15) it can be shown that the expected values of the product of $X_1$ and $X_3$ and the product of $X_2$ and $X_3$ are...

$$
E[X_1 X_3] = p_0 p_1 \\
E[X_2 X_3] = p_0 p_1
$$
(16)  (17)

**There Has Been One Prior Bond Default**

The equation for the conditional default correlation between $X_2$ and $X_3$ given that bond $B_1$ has already defaulted (i.e. $X_1 = 1$) is...

$$
\theta_{2,3} = \frac{E[X_2 X_3 | X_1 = 1] - E[X_2 | X_1 = 1] E[X_3 | X_1 = 1]}{\sqrt{Var(X_2 | X_1 = 1) Var(X_3 | X_1 = 1)}}
$$
(18)

The expected value of $X_2$ given that bond $B_1$ has already defaulted (uses Equation (14)) is...

$$
E[X_2 | X_1 = 1] = (0 \times P[X_2 = 0 | X_1 = 1]) + (1 \times P[X_2 = 1 | X_1 = 1]) \\
= P[X_2 = 1 | X_1 = 1] \\
= p_1
$$
(19)

The expected value of the square of $X_2$ given that bond $B_1$ has already defaulted (uses Equation (14)) is...

$$
E[X_2^2 | X_1 = 1] = (0^2 \times P[X_2 = 0 | X_1 = 1]) + (1^2 \times P[X_2 = 1 | X_1 = 1]) \\
= P[X_2 = 1 | X_1 = 1] \\
= p_1
$$
(20)

Using Equations (19) and (20) the variance of $X_2$ given that bond $B_1$ has already defaulted is...

$$
Var(X_2 | X_1 = 1) = E[X_2^2 | X_1 = 1] - (E[X_2 | X_1 = 1])^2 \\
= p_1 - p_1^2 \\
= p_1 (1 - p_1)
$$
(21)
The expected value of the product of $X_2$ and $X_3$ given that bond $B_1$ has already defaulted is...

$$
\mathbb{E}[X_2 X_3 | X_1 = 1] = (0 \times 0 \times P[X_2 = 0 \cap X_3 = 0 | X_1 = 1]) + (1 \times 0 \times P[X_2 = 1 \cap X_3 = 0 | X_1 = 1]) + (0 \times 1 \times P[X_2 = 0 \cap X_3 = 1 | X_1 = 1]) + (1 \times 1 \times P[X_2 = 1 \cap X_3 = 1 | X_1 = 1]) \\
= P[X_2 = 1 \cap X_3 = 1 | X_1 = 1] \\
= P[X_2 = 1 | X_1 = 1] P[X_3 = 1 | X_1 = 1, X_2 = 1]
$$

Note that per Equation (22) above the expected value of the product of $X_2$ and $X_3$ given that $X_1 = 1$ is the probability that bonds $B_2$ and $B_3$ both default given that bond $B_1$ has already defaulted. The equation for this compound probability is the probability that bond $B_2$ will default given that bond $B_1$ has already defaulted times the probability that bond $B_3$ will default given that both bonds $B_1$ and $B_2$ have already defaulted.

We can now equate equations Equations (18) and (22) and solve for the conditional probability that bond $B_3$ will default given that bonds $B_1$ and $B_2$ have already defaulted. The equation for the conditional probability that $X_3 = 1$ given that $X_1 = 1$ and $X_2 = 1$ is...

$$
P[X_2 = 1 | X_1 = 1, X_2 = 1] = \theta_{2,3} \sqrt{Var(X_2 | X_1 = 1)} Var(X_3 | X_1 = 1) + \mathbb{E}[X_2 | X_1 = 1] \mathbb{E}[X_3 | X_1 = 1] \\
p_3 P[X_3 = 1 | X_1 = 1, X_2 = 1] = \theta_{2,3} p_1 (1 - p_1) p_1 (1 - p_1) + p_1^2 \\
P[X_3 = 1 | X_1 = 1, X_2 = 1] = \theta_{2,3} (1 - p_1) + p_1
$$

For the sake of convenience we will define the variable $p_3$ to be the conditional probability that bond $B_3$ defaults given that bonds $B_1$ and $B_2$ have already defaulted. Using this definition we can rewrite Equation (23) above as...

$$
p_3 = P[X_3 = 1 | X_1 = 1, X_2 = 1] \\
= \theta_{2,3} (1 - p_1) + p_1 \\
= p_1 (1 - \theta_{2,3}) + \theta_{2,3}
$$

Using Equations (14) and (24) above the expected value of the product of $X_1$, $X_2$ and $X_3$ is...

$$
\mathbb{E}[X_1 X_2 X_3] = (0 \times 0 \times 0 \times P[X_1 = 0 \cap X_2 = 0 \cap X_3 = 0 | X_1 = 1]) + (1 \times 0 \times 0 \times P[X_1 = 1 \cap X_2 = 0 \cap X_3 = 0 | X_1 = 1]) + (1 \times 1 \times 0 \times P[X_1 = 1 \cap X_2 = 0 \cap X_3 = 1 | X_1 = 1]) + (0 \times 1 \times 1 \times P[X_1 = 0 \cap X_2 = 1 \cap X_3 = 1 | X_1 = 1]) + (0 \times 0 \times 1 \times P[X_1 = 0 \cap X_2 = 0 \cap X_3 = 1 | X_1 = 1]) + (1 \times 1 \times 1 \times P[X_1 = 1 \cap X_2 = 1 \cap X_3 = 1 | X_1 = 1]) \\
= P[X_1 = 1 \cap X_2 = 1 \cap X_3 = 1] \\
= P[X_1 = 1] P[X_2 = 1 | X_1 = 1] P[X_3 = 1 | X_1 = 1, X_2 = 1] \\
= p_0 p_1 p_2
$$

The Answer To Our Hypothetical Problem

Note that per Equations (3), (4) and (5) above the probability that $X_i = 1$ (i.e. Asset $A_i$ defaults) is $\mathbb{E}[X_i]$ and accordingly the probability that $X_i = 0$ (i.e. Asset $A_i$ does not default) is $1 - \mathbb{E}[X_i]$.

Question one to our problem is to find the probabilities that our bond portfolio will experience zero defaults, one default, two defaults and three defaults. Using Equations (3), (4), (5), (15), (16), (17) and (25) above the equation for the probability that our portfolio will experience zero defaults ($X_1 = 0, X_2 = 0, X_3 = 0$) is...

$$
P[0] = \mathbb{E}[(1 - X_1)(1 - X_2)(1 - X_3)] \\
= \mathbb{E}[(1 - X_1)(1 - X_2 - X_3 + X_2 X_3)] \\
= \mathbb{E}[1 - X_1 - X_2 - X_3 + X_1 X_2 + X_1 X_3 + X_2 X_3 - X_1 X_2 X_3] \\
= 1 - \mathbb{E}[X_1] - \mathbb{E}[X_2] - \mathbb{E}[X_3] + \mathbb{E}[X_1 X_2] + \mathbb{E}[X_1 X_3] + \mathbb{E}[X_2 X_3] - \mathbb{E}[X_1 X_2 X_3] \\
= 1 - p_0 - p_0 - p_0 + p_0 p_1 + p_0 p_1 + p_0 p_1 - p_0 p_1 p_2 \\
= 1 - 3 p_0 + 3 p_0 p_1 - p_0 p_1 p_2
$$
Using Equations (3), (15), (16) and (25) above the equation for the probability that our portfolio will experience one default \( \{X_1 = 1, X_2 = 0, X_3 = 0\} \) \( \ldots \text{or...} \ \{X_1 = 0, X_2 = 1, X_3 = 0\} \ \ldots \text{or...} \ \{X_1 = 0, X_2 = 0, X_3 = 1\} \) is...

\[
P[1] = E[X_1(1 - X_2)(1 - X_3)]
\]

\[
= E[X_1(1 - X_2 - X_3 + X_2X_3)]
\]

\[
= E[X_1 - X_1X_2 - X_1X_3 + X_1X_2X_3]
\]

\[
= E[X_1] - E[X_1X_2] - E[X_1X_3] + E[X_1X_2X_3]
\]

\[
= p_0 - p_0p_1 - p_0p_1 + p_0p_1p_2
\]

\[
= p_0 - 2p_0p_1 + p_0p_1p_2
\]

(27)

Using Equations (15) and (25) above the equation for the probability that our portfolio will experience two defaults \( \{X_1 = 1, X_2 = 1, X_3 = 0\} \ \ldots \text{or...} \ \{X_1 = 1, X_2 = 0, X_3 = 1\} \ \ldots \text{or...} \ \{X_1 = 0, X_2 = 1, X_3 = 1\} \) is...

\[
P[2] = E[X_1X_2(1 - X_3)]
\]

\[
= E[X_1X_2 - X_1X_2X_3]
\]

\[
= E[X_1X_2] - E[X_1X_2X_3]
\]

\[
= p_0p_1 - p_0p_1p_2
\]

(28)

Using Equation (25) above, the equation for the probability that our portfolio will experience three defaults \( X_1 = 1, X_2 = 1, X_3 = 1 \) is...

\[
P[3] = E[X_1X_2X_3]
\]

\[
= p_0p_1p_2
\]

(29)

Per Equations (1), (14) and (24) above, the values of \( p_0, p_1 \) and \( p_2 \) are...

\[
p_0 = 0.1000
\]

(30)

\[
p_1 = p_0(1 - \theta_{1,2}) + \theta_{1,2} = 0.10(1 - 0.30) + 0.30 = 0.3700
\]

(31)

\[
p_2 = p_1(1 - \theta_{2,3}) + \theta_{2,3} = 0.37(1 - 0.30) + 0.30 = 0.5590
\]

(32)

Given correlated defaults the answer to the problem is...

<table>
<thead>
<tr>
<th>Default</th>
<th>No Default</th>
<th>Combin</th>
<th>Indiv Prob</th>
<th>Total Prob</th>
<th>Payoff</th>
<th>Expected</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0.7903</td>
<td>0.7903</td>
<td>3,450,000</td>
<td>2,726,594</td>
<td>Equation (26)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0.0467</td>
<td>0.1400</td>
<td>2,700,000</td>
<td>378,132</td>
<td>Equation (27)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>0.0163</td>
<td>0.0490</td>
<td>1,950,000</td>
<td>95,454</td>
<td>Equation (28)</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0.0207</td>
<td>0.0207</td>
<td>1,200,000</td>
<td>24,820</td>
<td>Equation (29)</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>8</td>
<td></td>
<td>1.0000</td>
<td>3,225,000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Total expected return = \( \frac{3,225,000}{3,000,000} - 1 = 7.50\% \)

Note that in both the independent case and the correlated case the expected cash flow and total return of $3,225,000 and 7.50%, respectively, is the same in both cases. The difference is the shape of the probability distribution which in the correlated case puts much more weight on bad events (2 and 3 defaults) than does the independent case. The table below presents the respective probability distributions...

<table>
<thead>
<tr>
<th>Defaults</th>
<th>Independent</th>
<th>Correlated</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7290</td>
<td>0.7903</td>
</tr>
<tr>
<td>1</td>
<td>0.2430</td>
<td>0.1400</td>
</tr>
<tr>
<td>2</td>
<td>0.0270</td>
<td>0.0490</td>
</tr>
<tr>
<td>3</td>
<td>0.0010</td>
<td>0.0207</td>
</tr>
<tr>
<td>Total</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

To ignore correlation in credit risk management (and in other areas of valuation) is not only bad practice it is gross incompetence.