The Vasicek Interest Rate Process  
Part I - The Short Rate  

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The Vasicek interest rate model is a mathematical model that describes the evolution of the short rate of interest over time. The short rate is the annualized interest rate at which an entity can borrow money for an infinitesimally short period of time. Vasicek models the short rate as a Ornstein-Uhlenbeck process. An Ornstein-Uhlenbeck process is a mean-reverting process where the short rate is allowed to incorporate random shocks but is pulled back to its long-term mean whenever it moves away from it. Interest rates exhibit mean reversion, which is the tendency for a stochastic process to return over time to a long-term mean. Vasicek’s stochastic differential equation that describes the evolution of the short rate $R_t$ in continuous time is...

$$\delta R_t = \theta(\lambda - R_t) \delta t + \sigma \delta W_t \quad (1)$$

When the short rate moves below its long-term mean $\lambda$ the short rate drift becomes positive and the short rate is pulled upward. When the short rate moves above its long-term mean the short rate drift becomes negative and the short rate is pulled downward. The speed at which the drift is pulled upward of downward is given by the positive valued parameter $\theta$, which measures the speed of mean reversion. The greater the speed the faster the process reverts toward the long-term mean. Random shocks are introduced via the variables $\sigma$, which is the annualized short rate volatility, and $\delta W_t$, which is the change in the driving Brownian motion over the infinitesimally short time interval $[t, t + \delta t]$.

We will define the short rate at time period $t > s$ to be $R_t$, which is the short rate at period $s$ (known) plus the sum of the changes in the short rate from period $s$ to period $t$ (unknown). The equation for the short rate $R_t$ in continuous time is...

$$R_t = R_s + \int_s^t \delta R_u \quad (2)$$

In this white paper we will develop the mathematics of the Vasicek short rate stochastic process and use the mathematics to answer the three questions applicable to the hypothetical problem below...

**Our Hypothetical Problem**

We are tasked with pricing a pure-discount bond and for that task we need a short rate curve. Our go-forward interest rate assumptions are as follows...

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current short rate</td>
<td>$R_0$</td>
<td>0.04</td>
</tr>
<tr>
<td>Long-Term short rate mean</td>
<td>$R_\infty$</td>
<td>0.08</td>
</tr>
<tr>
<td>Mean reversion rate</td>
<td>$\theta$</td>
<td>0.40</td>
</tr>
<tr>
<td>Annualized short rate volatility</td>
<td>$\sigma$</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Question 1: Graph the short rate curve from year 0 thru year 10.

Question 2: What is the expected short rate at the end of years 1 and 3? (include confidence interval).

Question 3: What is the correlation between the short rate at the end of years 1 and 3?
The Vasicek Equation For The Short Rate Of Interest

We will define the function \( f(R_t, t) \) to be a function of time \( t \) and the short rate of interest at time \( t \). As we did in Equation (1) above we will define the variable \( \theta \) to be the rate of mean reversion, the variable \( R_t \) to be the short rate of interest at time \( t \), the variable \( \lambda \) to be the long-term short rate mean and the variable \( \sigma \) to be the annualized short rate volatility. Given these definitions we will further state that the equation for \( f(R_t, t) \) is...

\[
  f(R_t, t) = e^{\theta t}(R_t - \lambda)
\]  

(3)

The derivatives of Equation (3) with respect to time \( t \) and the short rate at time \( t \) are...

\[
  \frac{\delta f(R_t, t)}{\delta t} = \theta e^{\theta t}(R_t - \lambda) \quad \text{...and...} \quad \frac{\delta f(R_t, t)}{\delta R_t} = e^{\theta t} \quad \text{...and...} \quad \frac{\delta^2 f(R_t, t)}{\delta R_t^2} = 0
\]  

(4)

Per Ito’s Lemma the equation for the change in the stochastic short rate \( R_t \) over the infinitesimally small time interval \([t, t + \delta t]\) is...

\[
  \delta R_t = a(R_t, t)\delta t + b(R_t, t)\delta W_t
\]  

(5)

Using Equations (1) and (5) above, which both define the change in the short rate \( R_t \), and given that...

\[
  \delta R_t = a(R_t, t)\delta t + b(R_t, t)\delta W_t \quad \text{...and...} \quad \delta R_t = \theta(\lambda - R_t)\delta t + \sigma \delta W_t
\]

Then after mapping the two equations we can make the following definitions...

\[
  a(R_t, t) = \theta(\lambda - R_t) \quad \text{...and...} \quad b(R_t, t) = \sigma
\]  

(6)

Per Ito’s Lemma Equation (3) is once differentiable with respect to time \( t \) and twice differentiable with respect to the driving Brownian motion \( W_t \). Using a Taylor Series Expansion the equation for the change in \( f(R_t, t) \) is...

\[
  \delta f(R_t, t) = \left( \frac{\delta f(R_t, t)}{\delta t} + a(R_t, t)\frac{\delta f(R_t, t)}{\delta R_t} + \frac{1}{2} b(R_t, t)^2 \frac{\delta^2 f(R_t, t)}{\delta R_t^2} \right) \delta t + b(R_t, t)\frac{\delta f(R_t, t)}{\delta R_t} \delta W_t
\]  

(7)

Using Equations (4) and (6) above we can rewrite Equation (7) as...

\[
  \delta f(R_t, t) = (\theta e^{\theta t}(R_t - \lambda) + \theta e^{\theta t}(\lambda - R_t))\delta t + \sigma e^{\theta t}\delta W_t
\]

\[
  = \sigma e^{\theta t}\delta W_t
\]  

(8)

Taking integrals of both sides of Equation (8) above we get...

\[
  \int_{u=s}^{u=t} \delta f(R_u, u) = \int_{u=s}^{u=t} \sigma e^{\theta u}\delta W_u
\]

\[
  f(R_t, t) - f(R_s, s) = \sigma \int_{u=s}^{u=t} e^{\theta u}\delta W_u
\]  

(9)

After substituting Equation (3) into Equation (9) we can rewrite Equation (9) as...

\[
  e^{\theta t}(R_t - \lambda) - e^{\theta s}(R_s - \lambda) = \sigma \int_{u=s}^{u=t} e^{\theta u}\delta W_u
\]

\[
  e^{\theta t}R_t - e^{\theta t}\lambda - e^{\theta s}R_s + e^{\theta s}\lambda = \sigma \int_{u=s}^{u=t} e^{\theta u}\delta W_u
\]

\[
  e^{\theta t}R_t = e^{\theta s}R_s + e^{\theta t}\lambda - e^{\theta s}\lambda + \sigma \int_{u=s}^{u=t} e^{\theta u}\delta W_u
\]

\[
  R_t = e^{-\theta t}\left( e^{\theta s}R_s + e^{\theta t}\lambda - e^{\theta s}\lambda + \sigma \int_{u=s}^{u=t} e^{\theta u}\delta W_u \right)
\]  

(10)

Equation (10) is therefore the equation for the short rate at time \( t \) (\( R_t \)) given the short rate at time \( s \) (\( R_s \)), the rate of mean reversion (\( \lambda \)) and the annualized short rate volatility (\( \sigma \)).
Proof

We will now prove that short rate Equation (10) is the solution to Vasicek’s stochastic differential Equation (1) above. We will begin by making the following simplifying definitions...

\[ X = e^{-\theta t} \quad \text{and} \quad Y = e^{\theta s} R_s + e^{\theta t} \lambda - e^{\theta s} \lambda + \sigma \int_{u=s}^{u=t} e^{\theta u} \delta W_u \]  

(11)

Given the definitions in Equation (11) above we can rewrite Equation (10), which is the equation for the short rate at time \( t \), as...

\[ R_t = XY \]  

(12)

The derivatives of \( X \) are...

\[ \frac{\delta X}{\delta t} = -\theta e^{-\theta t} \quad \text{and} \quad \frac{\delta X}{\delta W_t} = 0 \quad \text{and} \quad \frac{\delta^2 X}{\delta W_t^2} = 0 \]  

(13)

The derivatives of \( Y \) are...

\[ \frac{\delta Y}{\delta t} = \theta \lambda e^{\theta t} \quad \text{and} \quad \frac{\delta Y}{\delta W_t} = \sigma e^{\theta t} \quad \text{and} \quad \frac{\delta^2 Y}{\delta W_t^2} = 0 \]  

(14)

Using a Taylor Series Expansion the equation for the change in the short rate \( R_t \) is...

\[ \delta R_t = \left( \frac{\delta X}{\delta t} Y + \frac{\delta Y}{\delta t} X \right) \delta t + \left( \frac{\delta X}{\delta W_t} Y + \frac{\delta Y}{\delta W_t} X \right) \delta W_t + \frac{1}{2} \left( \frac{\delta^2 X}{\delta W_t^2} Y + \frac{\delta^2 Y}{\delta W_t^2} X \right) \delta W_t^2 \]  

(15)

Using derivative Equations (13) and (14) above we can rewrite Equation (15) as

\[ \delta R_t = \left( -\theta e^{-\theta t} Y + \theta \lambda e^{\theta t} X \right) \delta t + \left( (0) Y + \sigma e^{\theta t} X \right) \delta W_t + \frac{1}{2} \left( (0) Y + (0) X \right) \delta W_t^2 \]

\[ = (-\theta R_t + \theta \lambda) \delta t + \sigma \delta W_t \]

\[ = \theta (\lambda - R_t) \delta t + \sigma \delta W_t \]  

(16)

Equation (16) is equivalent to Vasicek’s stochastic differential Equation (1), which concludes the proof.

Stochastic Short Mean And Variance

We can rewrite Equation (10), which is the equation for the stochastic short rate at time \( t \) given the short rate at time \( s \), as...

\[ R_t = R_s e^{-\theta (t-s)} + \lambda \left( 1 - e^{-\theta (t-s)} \right) + \sigma e^{-\theta t} \int_{u=s}^{u=t} e^{\theta u} \delta W_u \]  

(17)

Given that...

\[ \mathbb{E} \left[ \delta W_u \right] = 0 \]  

(18)

Using Equations (17) and (18) above the equation for the first moment of the distribution of \( R_t \) is...

\[ \mathbb{E} \left[ R_t \right] = \mathbb{E} \left[ R_s e^{-\theta (t-s)} + \lambda \left( 1 - e^{-\theta (t-s)} \right) + \sigma e^{-\theta t} \int_{u=s}^{u=t} e^{\theta u} \delta W_u \right] \]

\[ = R_s e^{-\theta (t-s)} + \lambda \left( 1 - e^{-\theta (t-s)} \right) + \sigma e^{-\theta t} \mathbb{E} \left[ \int_{u=s}^{u=t} e^{\theta u} \delta W_u \right] \]

\[ = R_s e^{-\theta (t-s)} + \lambda \left( 1 - e^{-\theta (t-s)} \right) \]

(19)

Using Equation (19) above we will make the following definition...

\[ \mu_t = \mathbb{E} \left[ R_t \right] = R_s e^{-\theta (t-s)} + \lambda \left( 1 - e^{-\theta (t-s)} \right) \]  

(20)
Using the definition in Equation (20) we can rewrite the stochastic short rate Equation (17) as...

\[ R_t = \mu_t + \sigma e^{-\theta t} \int_{u=s}^{u=t} e^{\theta u} \delta W_u \]

(21)

Given that...

When \( u \neq v \) then \( \mathbb{E} \left[ \delta W_u \delta W_v \right] = 0 \) ...and... When \( u = v \) then \( \mathbb{E} \left[ \delta W_u \delta W_v \right] = \mathbb{E} \left[ \delta W_u^2 \right] = \delta u \)  

(22)

Using the revised short rate Equation (21) and the expectations in Equations (18) and (22) the equation for the second moment of the distribution of \( R_t \), which is the expected value of the square of the short rate \( R_t \) given the short rate \( R_s \), is...

\[
\mathbb{E} \left[ R_t^2 \right] = \mathbb{E} \left[ \left( \mu_t + \sigma e^{-\theta t} \int_{u=s}^{u=t} e^{\theta u} \delta W_u \right)^2 \right]
\]

\[
= \left[ \mu_t^2 + 2 \mu_t \sigma e^{-\theta t} \int_{u=s}^{u=t} e^{\theta u} \delta W_u + \sigma^2 e^{-2\theta t} \int_{u=s}^{u=t} \int_{v=s}^{v=t} e^{\theta u} e^{\theta v} \delta W_u \delta W_v \right]
\]

\[
= \mu_t^2 + 2 \mu_t \sigma e^{-\theta t} \int_{u=s}^{u=t} e^{\theta u} \mathbb{E} \left[ \delta W_u \right] + \sigma^2 e^{-2\theta t} \int_{u=s}^{u=t} \int_{v=s}^{v=t} e^{\theta u} e^{\theta v} \mathbb{E} \left[ \delta W_u \delta W_v | u \neq v \right] + \sigma^2 e^{-2\theta t} \int_{u=s}^{u=t} \int_{v=s}^{v=t} e^{2\theta u} \mathbb{E} \left[ \delta W_u^2 | u = v \right]
\]

\[
= \mu_t^2 + 2 \mu_t \sigma e^{-\theta t} \int_{u=s}^{u=t} e^{\theta u} \mathbb{E} \left[ \delta W_u \right] + \sigma^2 e^{-2\theta t} \int_{u=s}^{u=t} \int_{v=s}^{v=t} e^{\theta u} e^{\theta v} \mathbb{E} \left[ \delta W_u \delta W_v | u \neq v \right] + \sigma^2 e^{-2\theta t} \int_{u=s}^{u=t} e^{2\theta u} \mathbb{E} \left[ \delta W_u^2 \right]
\]

\[
= \mu_t^2 + 2 \mu_t \sigma e^{-\theta t} \int_{u=s}^{u=t} e^{\theta u} \mathbb{E} \left[ \delta W_u \right] + \sigma^2 e^{-2\theta t} \int_{u=s}^{u=t} e^{2\theta u} \delta u
\]

(23)

After solving the integral in Equation (23) above the equation for the second moment of the distribution of \( R_t \) becomes...

\[
\mathbb{E} \left[ R_t^2 \right] = \mu_t^2 + \sigma^2 e^{-2\theta t} \int_{u=s}^{u=t} e^{2\theta u} \delta u
\]

\[
= \mu_t^2 + \sigma^2 e^{-2\theta t} \frac{1}{2\theta} \left[ e^{2\theta u} \right]_{s}^{t}
\]

\[
= \mu_t^2 + \sigma^2 e^{-2\theta t} \frac{1}{2\theta} \left( e^{2\theta t} - e^{2\theta s} \right)
\]

\[
= \mu_t^2 + \frac{\sigma^2}{2\theta} \left( 1 - e^{-2\theta(t-s)} \right)
\]

(24)

Using Equation (19) above the mean of the short rate at time \( t \) given the short rate at time \( s \) is...

\[
\text{mean} = \mathbb{E} \left[ R_t \right] = R_s e^{-\theta(t-s)} + \lambda \left( 1 - e^{-\theta(t-s)} \right)
\]

(25)

Using Equations (19), (20) and (24) above the variance of the short rate at time \( t \) given the short rate at time \( s \) is...

\[
\text{variance} = \mathbb{E} \left[ R_t^2 \right] - \left( \mathbb{E} \left[ R_t \right] \right)^2
\]

\[
= \mu_t^2 + \frac{\sigma^2}{2\theta} \left( 1 - e^{-2\theta(t-s)} \right) - \mu_t^2
\]

\[
= \frac{\sigma^2}{2\theta} \left( 1 - e^{-2\theta(t-s)} \right)
\]

(26)
Stochastic Short Rate Covariance

Imagine that we are currently sitting at time period 0 and that there are three future time periods \(x, y\) and \(z\) such that \(0 < x < y < z\). Given that \(R_x\) is the short rate at time \(x\) the equation for the short rates at time \(y (R_y)\) and at time \(z (R_z)\) can be written as...

\[
R_y = R_x + \int_{u=x}^{u=y} \delta R_u \quad \text{and...} \quad R_z = R_x + \int_{u=x}^{u=z} \delta R_u
\]  

(27)

Because \(y < z\) the increments in the driving Brownian motion during the time interval \([x, y]\) will be the same for short rates \(R_y\) and \(R_z\) and therefore the two short rates are positively correlated. The equation for the covariance between the short rate at time \(y\) and the short rate at time \(z\) is...

\[
\text{Covariance} \left[ R_y R_z \right] = E \left[ R_y R_z \right] - E \left[ R_y \right] E \left[ R_z \right]
\]  

(28)

Using Equation (21) above we can write the equations for the short rate at time \(y\) and at time \(z\) as...

\[
R_y = \mu_y + \sigma e^{-\theta y} \int_{u=x}^{u=y} e^{\theta u} \delta W_u \quad \text{and...} \quad R_z = \mu_z + \sigma e^{-\theta z} \int_{u=x}^{u=z} e^{\theta u} \delta W_u
\]  

(29)

Per covariance Equation (28) we need the expectation of the product of the two short rates \(R_y\) and \(R_z\). This expectation in equation form is...

\[
E \left[ R_y R_z \right] = E \left[ \mu_y e^{-\theta y} \int_{u=x}^{u=y} e^{\theta u} \delta W_u \right] \left[ \mu_z e^{-\theta z} \int_{u=x}^{u=z} e^{\theta u} \delta W_u \right]
\]  

(30)

Given the expectation in Equation (18) above Equation (30) becomes...

\[
E \left[ R_y R_z \right] = \mu_y \mu_z + \sigma^2 e^{-\theta(y+z)} \int_{u=x}^{u=y} \int_{v=x}^{v=z} e^{\theta u} e^{\theta v} E \left[ \delta W_u \delta W_v \right]
\]  

(31)

We can rewrite the double integral in Equation (31) as...

\[
\int_{u=x}^{u=y} \int_{v=x}^{v=z} e^{\theta u} e^{\theta v} E \left[ \delta W_u \delta W_v \right] = \int_{u=x}^{u=y} \int_{v=x}^{v=z} e^{\theta u} e^{\theta v} E \left[ \delta W_u \delta W_v \right] \quad u \neq v + \int_{u=x}^{u=y} \int_{v=x}^{v=z} e^{\theta u} e^{\theta v} E \left[ \delta W_u \delta W_v \right] \quad u = v
\]  

(32)

Given the expectations in Equation (22) above Equation (31) becomes...

\[
E \left[ R_y R_z \right] = \mu_y \mu_z + \sigma^2 e^{-\theta(y+z)} \int_{u=x}^{u=y+z} e^{\theta u} \delta u
\]  

(33)

\[
= \mu_y \mu_z + \sigma^2 e^{-\theta(y+z)} \left( \frac{1}{2\theta} \right) (e^{2\theta(y+z)} - e^{2\theta x})
\]
Using Equations (28) and (33) above we can write the covariance of the short rate at time \( y \) and the short rate at time \( z \) given the short rate at time \( x \) as...

\[
\text{Covariance } (R_y R_z) = \text{E} \left[ R_y R_z \right] - \text{E} \left[ R_y \right] \text{E} \left[ R_z \right] \\
= \mu_y \mu_z + \sigma^2 e^{-\theta(y+z)} \frac{1}{2 \theta} \left( e^{2 \theta (y+z)} - e^{2 \theta x} \right) - \mu_y \mu_z \\
= \sigma^2 e^{-\theta(y+z)} \left( e^{2 \theta (y+z)} - e^{2 \theta x} \right) \\
\]

(34)

Simulating The Stochastic Short Rate

Given that \( R_t \) is the stochastic short rate at time \( t \) (unknown) given the short rate at time \( s < t \) (known), \( m \) is the short rate mean per Equation (25) and \( \nu \) is the short rate variance per Equation (26), we can define the normalized random variable \( Z \) via the following equation...

\[
\frac{R_t - m}{\sqrt{\nu}} = Z \quad \text{such that... } Z \sim N \left[ 0, 1 \right] \\
\]

(35)

By rearranging Equation (35) above the simulated value of the short rate at some future time \( t \) given the short rate at time \( s < t \) is...

\[
R_t = m + \sqrt{\nu} Z \quad \text{where... } Z \sim N \left[ 0, 1 \right] \\
\]

(36)

Answers To Our Hypothetical Problem

Question 1: Graph the spot rate curve from year 0 thru year 10 - Uses Equation (25) above.

[Spot Rate Curve graph]

Question 2: What is the expected short rate at the end of years 1 and 3? (include confidence interval) - Uses Equations (25) and (26) above.

The expected spot rate at the end of year 1 is...

\[
\text{mean} = R_0 \exp(-\theta(1-0)) + \lambda \left( 1 - \exp(-\theta(1-0)) \right) \\
= 0.04 \times \exp(-0.40 \times 1) + 0.08 \times (1 - \exp(-0.40 \times 1)) \\
= 0.032 \\
\]

(37)
The expected spot rate at the end of year 3 is...

\[
\text{mean} = R_0 e^{-\theta(3-0)} + \lambda \left(1 - e^{-\theta(3-0)}\right) \\
= 0.04 \times \exp(-0.40 \times 3) + 0.08 \times (1 - \exp(-0.40 \times 3)) \\
= 0.068
\] (38)

The spot rate variance at the end of year 1 is...

\[
\text{variance} = \frac{\sigma^2}{2\theta} \left(1 - e^{-2\theta(1-0)}\right) \\
= \frac{0.02^2}{(2)(0.40)} \times (1 - \exp(-2 \times 0.40 \times 1)) \\
= 0.000275
\] (39)

The spot rate variance at the end of year 3 is...

\[
\text{variance} = \frac{\sigma^2}{2\theta} \left(1 - e^{-2\theta(3-0)}\right) \\
= \frac{0.02^2}{(2)(0.40)} \times (1 - \exp(-2 \times 0.40 \times 3)) \\
= 0.000455
\] (40)

The confidence interval (at 2 std dev) is...

Spot rate end of year 1 = 0.032 ± 2 × \sqrt{0.000275} = 0.032 ± 0.0332 (41)
Spot rate end of year 3 = 0.068 ± 2 × \sqrt{0.000455} = 0.068 ± 0.0426 (42)

Question 3: What is the correlation between the short rate at the end of year 1 and year 3? - Uses Equations (26) and (34) above.

The covariance between the spot rate at the end of year 1 and year 3 is...

\[
\text{Covariance } (R_y R_z) = \sigma^2 e^{-\theta(1+3)} \frac{1}{2\theta} \left(e^{2\theta(1\wedge3)} - e^{2\theta(0)}\right) \\
= 0.02^2 \times e^{-0.40\times4} \div (2 \times 0.40) \times \left(e^{2\times0.40\times1} - 1\right) \\
= 0.000124
\] (43)

Using the variance of the spot rate at the end of years 1 and 3 as calculated in Question 2 above the correlation between the spot rate at the end of year 1 and year 3 is...

\[
\text{Correl } (R_1, R_3) = \frac{\text{Covariance } (R_1 R_3)}{\text{Sdev } R_1 \times \text{Sdev } R_3} = \frac{0.000124}{\sqrt{0.000275} \times \sqrt{0.000455}} = 0.35
\] (44)