Schurman XGGM Model - Extended Gordon Growth Model in Continuous Time

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The Gordon Growth Model (GGM) is a model for determining the intrinsic value of a stock based on a future stream of dividends that grow at a constant rate. The equation for stock price at time zero is the solution to an infinite series of dividend payments in discrete time. The equation for stock price at time zero via the standard GGM is...

$$S_0 = \frac{D_1}{k - g} \tag{1}$$

The dependent variable S_0 in the equation above is stock price at time zero (now). The independent variables D_1 is the expected dividend per share one year from now, g is the annual dividend growth rate (in perpetuity), and k is the annual required rate of return for an equity investor (i.e. the discount rate).

We will define dividends at time t to be free cash flow at time t, which is heretofore defined as revenue minus variable costs minus fixed costs minus capital expenditures plus the debt tax shield. The Extended Gordon Growth Model (XGGM) will be used to calculate enterprise value, which means that fixed costs will not include interest expense on debt or dividends on preferred stock. Dividends in this context will be free cash flow available for distribution to all security holders (debt, preferred stock, common stock, etc.). We will extend the GGM such that the components of free cash flow (revenue contribution, fixed costs, capital expenditures and the debt tax shield) are valued individually and in continuous time. These extensions to the original model will give us added flexibility in our valuation.

Revenue Contribution

Given that R_t is annualized revenue at time t (in years) and μ is the revenue growth rate, we will define the change in annualized revenue to be in accordance with the following ODE (ordinary differential equation)...

$$\delta\, R_t = \mu\, R_t\, \delta t \tag{2}$$

The solution to the ODE in Equation (2) above is the equation for annualized revenue at time $t > 0$ as a function of annualized revenue at time zero, which is...

$$R_t = R_0\, e^{\mu t} \tag{3}$$

We will define the variable $\bar{R}_t$ to be revenue received over the infinitesimally small time interval $[t, t + \delta t]$. Using annualized revenue Equation (3) above the equation for $\bar{R}_t$ is...

$$\bar{R}_t = R_t\, \delta t = R_0\, e^{\mu t}\, \delta t \tag{4}$$

We will define revenue contribution as revenue minus variable costs. Variable costs would include costs that are truly variable such as direct labor, raw materials, etc. In theory variable costs would go to zero if the revenue stream did likewise. We will define the variable θ (contribution margin) to be one minus the ratio of variable costs to revenue. Using Equation (4) above the equation for revenue contribution received over the time interval $[t, t + \delta t]$ is...

$$\theta\, \bar{R}_t = \theta\, R_0\, e^{\mu t}\, \delta t \tag{5}$$

Note that depreciation expense is a variable cost and is therefore included in theta (θ). Even though depreciation is a non-cash expense we will convert this non-cash expense into a cash expense by assuming that there is one dollar of assets purchased for every dollar of assets written off via depreciation (i.e. the loss in value of the asset base via depreciation is immediately replaced).

Fixed Costs

The GGM assumes that all expenses are variable. If total costs as a percent of revenue are expected to remain constant then modeling fixed costs separate from revenue contribution yields very little if any marginal benefit. If this ratio is expected to change over time then modeling fixed costs is not only recommended but necessary. We will assume that fixed costs are not a function of revenue and grow at the rate of inflation. Given that F_t is annualized fixed costs at time t and ι is the rate of inflation, we will define the change in annualized fixed costs to be in accordance with the following ODE...

$$\delta\, F_t = \iota\, F_t\, \delta t \tag{6}$$

The solution to the ODE in Equation (6) above is the equation for annualized fixed costs at time t as a function of annualized fixed costs at time zero, which is...

$$F_t = F_0\, e^{\iota t} \tag{7}$$

We will define the variable $\bar{F}_t$ to be fixed costs incurred over the infinitesimally small time interval $[t, t + \delta t]$. Using annualized fixed costs Equation (7) above the equation for $\bar{F}_t$ is...

$$\bar{F}_t = F_t\, \delta t = F_0\, e^{\iota t}\, \delta t \tag{8}$$

Economies of Scale

As noted above the standard GGM assumes that all costs are variable. A case may arise where a company's contribution margin will increase over time as a function of economies of scale. In other words, a company's current revenue contribution includes an inordinate amount of fixed costs such that the ratio of fixed costs to revenue will decrease over time and eventually become insignificant as the company's revenues grow. Given that the contribution margin at time $t = 0$, which includes fixed and variable costs, is θ_0, and the contribution margin at some future time $t = \infty$, where the ratio of fixed costs to revenue is near zero such that all costs are variable, is θ_∞, then we can model the company's operating expense as follows...

$$R_0\, (1 - \theta_0) = F_0 + R_0\, (1 - \theta_e \infty) \tag{9}$$

In Equation (9) above the company's current operating expense ($R_0\, \theta_0$) is comprised of fixed costs of F_0 and variable costs of $R_0\, \theta_\infty$. When building our valuation equation we want the option to use a contribution margin of θ_0 and fixed costs of zero or a contribution margin of θ_∞ and fixed costs of F_0. If fixed costs grow at the rate of inflation then it is a mathematical certainty that if the revenue growth rate is greater than the rate of inflation then the company's contribution margin will increase to θ_∞ over time. If we can estimate the value of θ_∞ then we can solve for F_0 in Equation (9) as follows...

$$F_0 = R_0\, (\theta_\infty - \theta_0) \tag{10}$$

If our valuation model uses the fixed costs that we calculated in Equation (10) above and a contribution margin equal to θ_∞ then contribution margin will approach θ_∞ as time goes to infinity. Given that V_t is annualized variable costs at time t, F_t is annualized fixed costs at time t, R_t is annualized revenue at time t and the revenue growth rate μ is greater than the inflation rate ι, then the following equation will hold...

$$\lim_{t \to \infty} \frac{F_t + V_t}{R_t} = \theta_\infty \tag{11}$$

Note that the greater the difference between the revenue growth rate and the rate of inflation the sooner this convergence will take place.

Capital Expenditures

Given that A_t is total assets at time t, R_t is annualized revenue at time t and ϕ is the target ratio of total assets to annualized revenue, we will define the equation for total assets at time t to be...

$$A_t = \phi\, R_t \tag{12}$$

After substituting annualized revenue Equation (3) into total assets Equation (12) above the equation for total assets at time t as a function of annualized revenue at time zero is...

$$A_t = \phi\, R_0\, e^{\mu t} \tag{13}$$

Note that the first derivative of Equation (13) with respect to time is...

$$\frac{\delta A_t}{\delta t} = \phi\,\mu\,R_0\,e^{\mu t} \tag{14}$$

If revenue is expected to grow over time then total assets must also grow so as to maintain the target ratio. A growth in assets means that we must provide for capital expenditures in our model. If we define capital expenditures at time t ($\bar{X}_t$) to be the change in total assets over the infintesimally small time interval $[t, t+\delta t]$, then the equation for capital expenditures over the time interval $[t, t+\delta t]$ is...

$$\bar{X}_t = \delta A_t \tag{15}$$

After substituting derivative Equation (14) into Equation (15) above the equation for capital expenditures over the time interval $[t, t+\delta t]$ becomes...

$$\bar{X}_t = \phi\,\mu\,R_0\,e^{\mu t}\,\delta t \tag{16}$$

Debt Tax Shield

Given that dividend payments are not tax deductible and debt interest payments are the company may be better off if it uses a mix of debt and equity rather than just equity in it's capital structure. Cash flow applicable to the debt tax shield consists of income tax refunds generated by deducting debt interest payments on the company's tax return. Given that R_t is annualized revenue at time t, ω is the target ratio of debt to annualized revenue, α is the debt interest rate and λ is the income tax rate, the equation for the debt tax shield ($\bar{Q}_t$) over the infintesimally small time interval $[t, t+\delta t]$ is...

$$\bar{Q}_t = R_t\,\omega\,\alpha\,\lambda\,\delta t \tag{17}$$

After substituting the R_t in Equation (17) above with annualized revenue Equation (3) the equation for the debt tax shield over the time interval $[t,\ t+\delta t]$ becomes...

$$\bar{Q}_t = R_0\,e^{\mu t}\,\omega\,\alpha\,\lambda\,\delta t \tag{18}$$

Note that the value of the debt tax shield is an after-tax number.

Cash Flow

We will define cash flow as after-tax revenue contribution minus after-tax fixed costs minus capital expenditures plus the debt tax shield. Given that λ is the income tax rate and the equations presented above, the equation for cash flow (C_t) over the infinitesimally small time interval $[t, t+\delta t]$ is...

$$C_t = \left[\theta\,\bar{R}_t - \bar{F}_t\right](1-\lambda) - \bar{X}_t + \bar{Q}_t \tag{19}$$

After replacing the variables $\theta\bar{R}_t$, $\bar{F}_t$, $\bar{X}_t$ and $\bar{Q}_t$ in Equation (19) above with Equations (5), (8), (16) and (18), respectively, the equation for cash flow over the time interval $[t, t+\delta t]$ becomes...

$$C_t = \left[\theta\,R_0\,e^{\mu t}\,\delta t - F_0\,e^{\iota t}\,\delta t\right](1-\lambda) - \phi\,\mu\,R_0\,e^{\mu t}\,\delta t + R_0\,e^{\mu t}\,\omega\,\alpha\,\lambda\,\delta t \tag{20}$$

Cash flow Equation (20) above simplifies to...

$$C_t = R_0\,e^{\mu t}\left[\theta(1-\lambda) - \phi\,\mu + \omega\,\alpha\,\lambda\right]\delta t - (1-\lambda)\,F_0\,e^{\iota t}\,\delta t \tag{21}$$

Present Value of Cash Flow

The present value of cash flow modeled as a perpetuity would be the integral version of Equation (21) with the upper bound of integration equal to infinity and discounted at the discount rate. The equation for the present value of cash flow (V_0) is therefore...

$$
\begin{aligned}
V_0 &= \int_0^\infty C_t \, e^{-\kappa t} \\
&= \int_0^\infty \left\{ R_0 e^{\mu t} \left[\theta(1-\lambda) - \phi\mu + \omega\alpha\lambda \right] \delta t - (1-\lambda) F_0 e^{\iota t} \, \delta t \right\} e^{-\kappa t} \\
&= \int_0^\infty R_0 e^{(\mu-\kappa)t} \left[\theta(1-\lambda) - \phi\mu + \omega\alpha\lambda \right] \delta t - \int_0^\infty (1-\lambda) F_0 e^{(\iota-\kappa)t} \, \delta t \\
&= R_0 \left[\theta(1-\lambda) - \phi\mu + \omega\alpha\lambda \right] \int_0^\infty e^{(\mu-\kappa)t} \delta t - (1-\lambda) F_0 \int_0^\infty e^{(\iota-\kappa)t} \delta t
\end{aligned} \tag{22}
$$

The solution to the first integral in equation (22) is...

$$
\begin{aligned}
V_A &= \int_0^\infty e^{(\mu-\kappa)t} \delta t \\
&= \frac{1}{\mu-\kappa} \left[e^{(\mu-\kappa)\times\infty} - e^{(\mu-\kappa)\times 0} \right] \\
&= \frac{1}{\mu-\kappa} \left[0 - 1 \right] \\
&= \frac{1}{\kappa-\mu}
\end{aligned} \tag{23}
$$

Note that because μ is less than κ the limit of $e^{(\mu-\kappa)\times\infty}$ is zero.

The solution to the second integral in equation (22) using the logic in equation (23) above and the fact that ι is less than κ is...

$$
V_B = \int_0^\infty e^{(\iota-\kappa)t} \, \delta t = \frac{1}{\kappa-\iota} \tag{24}
$$

After solving the two integrals in equation (22) the equation for enterprise value becomes...

$$
\begin{aligned}
V_0 &= R_0[\theta(1-\lambda) - \phi\mu + \omega\alpha\lambda] \times V_A - (1-\lambda) F_0 \times V_B \\
&= \frac{R_0[\theta(1-\lambda) - \phi\mu + \omega\alpha\lambda]}{\kappa-\mu} - \frac{F_0(1-\lambda)}{\kappa-\iota}
\end{aligned} \tag{25}
$$

A Hypothetical Case

We are tasked with estimating the enterprise value of Company ABC given the following parameters:

Parameters to the Problem:

Description	Value	Model Parameter Value
Current annualized revenue	1000	$R_0 = 1000$
Current annualized fixed costs	300	$F_0 = 300$
Annualized revenue growth rate	4%	$\mu = \ln(1+0.04) = 0.0392$
Annualized fixed costs inflation rate	3%	$\iota = \ln(1+0.03) = 0.0296$
Annualized rate paid on debt	8%	$\alpha = \ln(1+0.08) = 0.0770$
Annualized discount rate	20%	$\kappa = \ln(1+0.20) = 0.1823$
Contribution margin	60%	$\theta = 0.60$
Income tax rate	35%	$\lambda = 0.35$
Target assets to annualized revenue ratio	25%	$\phi = 0.25$
Target debt to annualized revenue ratio	30%	$\omega = 0.35$

Problem solution using equation (25) **:**

$$
\begin{aligned}
V_0 &= \frac{R_0[\theta(1-\lambda) - \phi\mu + \omega\alpha\lambda]}{\kappa - \mu} - \frac{F_0(1-\lambda)}{\kappa - \iota} \\
&= \frac{1000[0.60(1-0.35) - 0.0392 \times 0.25 + 0.30 \times 0.0770 \times 0.35]}{0.1823 - 0.0392} - \frac{300 \times (1-0.35)}{0.1823 - 0.0296} \\
&= 1437 \qquad (26)
\end{aligned}
$$